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Bounded Topology

A convenient foundation for Topology

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Within Topology a lot kind of structures are being studied and examined.
Following this idea some "classical" of them such as topological spaces, uniform spaces, proximity spaces, contiguity spaces, Cauchy-spaces etc. then were discussed in the realm of more general spaces with additional, particular more convenient properties, not covered by the classical ones.
So, the concepts of nearness spaces, convergence spaces, syntopogeneous spaces are being established containing the above mentioned spaces in a nice manner and moreover lead us to extended theorems concerning completions, compactifications or topological extensions as well.
Another interesting concept was developed by Doìtchinóv, who introduced supertopological spaces in order to unify topological, proximal and uniform spaces again. The basic notion of a supertopology is given by a corresponding function over a set of bounded sets (B-sets) on a set X which naturally assigns a neighbourhood-system to each bounded set of X. "Continuity" of maps is then defined in an obvious way leading us in special cases to "delta-maps" or "continuous functions" , respectively.
The resulting category STOP contains TOP and PROX by simple variations of the B-set to { 0 }u { {x } : x e X } or P X , respectively.
Then Doìtchinóv proves a certain relationship of some special classes of supertopologies - called b-supertopologies – with compactly determined extensions.
But, topological extensions are closely related to nearness structures of various kind , too.
So, the question raises whether there exists a common concept of nearness- and super-topological spaces in which the different theorems could be expressed by only one single statement.
Thus, the new category SN of supernearness spaces, now denoted by b-NEAR and corresponding maps was introduced by the author in 2002 naturally solving the above mentioned problems. In this context the reader is referred to TOPOLOGY ATLAS Invited Contributions vol.8, no3 (2003).
Again, in the past constructs of various "convergence types" were considered in order to discover more "convenient" categories besides the classical ones of topological or uniform spaces. In one direction, the realm of Convenient Topology, strong topological universes were studied, i.e. concrete categories where initial structures exist, fibres are small, and which satisfy a terminal separator property. Consequently , natural function

spaces exists in such categories (i.e. they are cartesian closed), quotients are stable under products, and in addition such categories are extensional.
Moreover, a certain symmetry was proposed, leading to symmetric convergence structures, together with various generalizations of symmetric topological structures, as well as to uniform convergence structures and various generalizations to uniform structures. Among them the nearness spaces, merotopic spaces and Cauchy spaces seem to be of great interest.
In a second direction , referred to "Non-symmetric Convenient Topology" by Preuss, strong topological universes are available, in which non-symmetric convergence structures, such as topological structures and their various generalizations, e.g. limit spaces, pseudotopological spaces as well as set-convergence spaces and also supernearness spaces play an important role. Moreover, uniform convergence structures such as quasiuniformities and various generalizations can be dealt with.
In both cases, all the universes considered can easily be described by means of suitable axioms. Now having the corresponding constructs, some nice properties arising from the classical ones, like compactness or completeness, are described in order to obtain a general "compactification theory" or a "completion theory", respectively.
Moreover, in some cases a comprehensive "extension theory" was created in order to describe both processes of compactification and completion in common terms.
On the other hand , if a topological construct fails to have certain convenient properties, e.g. being cartesian closed or extensional, respectively, it is often possible to embed the given topological construct in a new one with the desired properties.
The minimal such extensions will be called the corresponding "hulls".
So, by construction, if the topological universe hull of a construct $\underline{C}$ exists, it is the smallest topological universe $\underline{U}$ in which $\underline{C}$ is finally dense.
For example, the topological universe hull of TOP turns out to be the construct PSTOP of pseudotopological spaces introduced by Choquet in 1948. The topological universe hull of the construct STOP of supertopological spaces was determined in 1989 by Wyler to be the construct of "Choquet set-convergence spaces".
By bringing together set-convergence spaces and preuniform convergence spaces in the sense of Preuss, we fill the gap between them by introducing a new category of so-called "b-convergence spaces" . As a basic concept we consider uniform filters converging to bounded subsets. Thus, in special cases, we recover the constructs of set-convergence spaces (Choquet set-convergence spaces) and preuniform convergence spaces (semiuniform convergence spaces), respectively. This now enables us to simultaneously express generalized "topological" and "uniform" aspects by common means, but, as pointed out above, with respect to the branches of Convenient Topology and Non-symmetric Convenient Topology as well. The resulting category b-CONV is topological.
So in general , subspaces and products , or quotients and sums as well are simultaneously formed by supplying the corresponding sets with the initial (respectively final) b-convergence with respect to the given data. Moreover, we will claim that *pointed* b-convergence leads us to a strong topological universe in which the constructs TOP and UNIF both can be embedded in particulary nice fashion. Besides, we have that "topological extensions" are closely related to corresponding b-convergences.
Well-known "topological extensions" in the literature are the Smirnov-compactification of an Efremovic proximity space, or the T1-extension related to a Lodato proximity space, or, more generally, the "Herrlich-Bentley"-extension of a so called "bunch-determined" nearness space.
All these constructions on a nearness structure may be viewed as special cases of a more *general* theory of topological extensions and their related b-convergence.

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At last returning to the concept of syntopogeneous spaces introduced by Császár in 1963, here, we describe commoness with supertopological spaces, too.
First, we note that the classical structures are given by a set of relations on the powerset of a set X satisfying certain axioms. As done before we replace PX by an arbitrary B-set ***Bx*** on X and define a b-*topogeneous* order (on ***Bx***) as a subset < c ***Bx*** x PX equipped with the following properties:

(b-top1) 0 < 0;
(b-top2) B e ***Bx*** implies B < X;
(b-top3) B < A implies B c A;
(b-top4) B c B* < A* c A imply B < A;
(b-top5) B < A1, A2 imply B < A1 n A2.

A b-topogeneous order < is then called

(i) *additive* iff B1 < A , B2 < A and B1u B2 e ***Bx*** imply B1 u B2 < A;
(ii) *linked* iff B < A1 and B < A2 imply B < A1 u A2;
(iii) *screened* B1, B2 < A imply B1 n B2 < A.

We will call a b-topogeneous order “<” *full b-topogeneous* iff it satisfies (i) through (iii).
In the case that ***Bx*** is *saturated* which means we have X e ***Bx***, then the definitions of a topogeneous order in the sense of Csàszàr and a full b-topogeneous order coincide.
Note, that in this case ***Bx*** equals PX.
Next, we define the *square* of a b-topogeneous order < by setting:
B <** A iff there exists A* e PX with B < A* and B* < A for each B* e ***Bx*** with B* c A*.
Having this we call a set S of b-topogeneous orders a *b-syntopogeneous* structure (on ***Bx***) iff S satisfies the following conditions:
(b-syn1) S is not empty;
(b-syn2) <1 , <2 e S imply there exists <3 e S such that <1 u <2 c <3;
(b-syn3) <1 e S implies that <1 c <** for some < e S.
In this context S is called a *b-topogenous* space iff Card S = 1.
Hence, a b-topogeneous order “<” “generates” a b-topogeneous space S = { < } iff “<” is *squared* which means that the equation < = <** holds.
Moreover, let us call a function f from a b-syntopogeneous space (X, ***Bx***, Sx) into a space (Y, ***By***, Sy) *syncontinuous* iff f is *bounded* which means that { f [B] : B e ***Bx*** } c ***By***, and it additionally satisfies the following condition:
For each <y e Sy there exists <x e Sx so that B <y A implies Bx <x f’ [A] for each Bx e ***Bx*** with Bx c f’ [B] (f’ […] denotes the inverse image of “ …”).
At last, we call a b-syntopogeneous space (X, ***Bx***, S) *saturated* iff ***Bx*** is saturated, hence the syntopogeneous spaces in the sense of Csàszàr coincide with the saturated b-syntopogeneous spaces in which all b-topogeneous orders are full b-topogeneous.
Additionally we have that s-continuous functions between syntopogeneous spaces are exactly the syncontinuous maps between the corresponding b-syntopogeneous spaces!
On the other hand each supertopological space (X, ***Mx***, #) naturally induces a b-topogeneous space (X, ***Mx***, S#) with S# = { <# } by setting :
B <# A iff A e #(B) . Conversely, we assign to each B e ***Bx*** the neighborhoodsystem N< (B) : = { U c X : B < U}.

Then, a function f between supertopological spaces (X, ***Mx***, #) and (Y, ***My***, +) is continuous, i.e. f is bounded and additionally holds
B e ***Mx*** and V e +(f [B]) implies f' [V] e # (B), iff
f is syncontinuous between the corresponding b-syntopogeneous spaces.
These facts then induce an isomorphism between STOP and the category sb-SYN of *simple* b-syntopogeneous spaces and syncontinuous maps.
By the way, a b-syntopogeneous space (X, ***Bx***, S) is called *simple* iff Card S = 1.
Now, we conclude that the above mentioned "b-categories" seems to be a first step to a **new** foundation for TOPOLOGY which we called **BOUNDED TOPOLOGY**.

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